Intro. to ODEs

Quiz 2 **Solutions**

1) Find the general solution to the differential equation $\frac{dy}{dx} = \frac{\cos(x)(1+y^2)}{y}$.

This is a separable first order differential equation.

$$\begin{aligned}
\int \frac{y}{1+y^2}\,dy &= \int \cos(x)\,dx + C \\
\frac{1}{2}\ln(1+y^2) &= \sin(x) + C \\
\ln(1+y^2) &= 2\sin(x) + C \\
1+y^2 &= Ce^{2\sin(x)} \\
y(x) &= \pm\sqrt{Ce^{2\sin(x)} - 1}
\end{aligned}$$

2) Solve the following initial value problem.

$$\begin{aligned}
x\frac{dy}{dx} + 2y &= x^3 \qquad (x > 0) \\
y(1) &= 0
\end{aligned}$$

This is a first order linear differential equation. The first thing is to get the equation into the proper form (by dividing by x).

$$\frac{dy}{dx} + \frac{2}{x}y = x^2$$

The correct integrating factor is

$$\mu(x) = e^{\int 2/x\,dx} = e^{2\ln(x)} = x^2.$$

So, the general solution is

$$y(x) = \frac{\int x^2 \cdot x^2\,dx + C}{x^2} = \frac{1}{5}x^3 + \frac{C}{x^2}.$$

Fitting the initial datum gives

$$y(1) = \frac{1}{5} + C = 0 \longrightarrow C = -\frac{1}{5}.$$

So, the solution is

$$y(x) = \frac{x^3}{5} - \frac{1}{5x^2}.$$

3) Solve the following initial value problem.

$$\frac{dy}{dx} - \frac{3}{x}y = x^3 \cos(x)\sin(x) \qquad (x > 0)$$
$$y(\pi) = 1$$

This is a first order linear differential equation. The integrating factor is

$$\mu(x) = e^{\int -3/x \, dx} = e^{-3\ln(x)} = \frac{1}{x^3}.$$

This means the general solution is given by

$$\begin{aligned} y(x) &= \frac{\int x^3 \cos(x)\sin(x) \cdot \frac{1}{x^3} \, dx + C}{\frac{1}{x^3}} \\ &= x^3 \left(\int \cos(x)\sin(x) \, dx + C \right) \\ &= \frac{1}{2}x^3 \sin^2(x) + Cx^3. \end{aligned}$$

Fitting the initial datum gives

$$y(\pi) = \frac{1}{2}\pi^3 \sin^2(\pi) + C\pi^3 = C\pi^3 = 1 \longrightarrow C = \frac{1}{\pi^3}.$$

So, the solution to the initial value problem is

$$y(x) = \frac{1}{2}x^3 \sin^2(x) + \left(\frac{x}{\pi}\right)^3$$

4) Find the general solution to the differential equation $\frac{dy}{dx} = \frac{xe^x}{y\sqrt{1+y^2}}$.

This is a seperable first order differential equation.

$$\begin{aligned} \int y\sqrt{1+y^2} \, dy &= \int xe^x \, dx + C \\ \frac{1}{3}\left(1+y^2\right)^{3/2} &= xe^x - e^x + C \\ \left(1+y^2\right)^{3/2} &= 3xe^x - 3e^x + C \\ 1 + y^2 &= (3xe^x - 3e^x + C)^{2/3} \\ y^2 &= [3(x-1)e^x + C]^{2/3} - 1 \\ y(x) &= \pm\sqrt{[3(x-1)e^x + C]^{2/3} - 1} \end{aligned}$$